

Principal bundles of embeddings and nonlinear Grassmannians

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Abstract

We present several principal bundles of embeddings of compact manifolds (with or without boundary) whose base manifolds are nonlinear Grassmannians. We study their infinite dimensional differential manifold structure in the Fréchet category. This study is motivated by the occurrence of such objects in the geometric Lagrangian formulation of free boundary continuum mechanics and in the study of the associated infinite dimensional dual pairs structures.

1 Introduction

In this paper, we consider several infinite dimensional principal bundles whose total space is a set of embeddings and whose structure group is given by a group of reparametrizing diffeomorphisms. The base space of these principal bundles are certain classes of nonlinear Grassmannians, that is, sets of submanifolds. Such infinite dimensional objects appear naturally, for example, in the geometric Lagrangian and Hamiltonian formulations of free boundary continuum mechanics and the associated processes of reduction by symmetries, see e.g., [Lewis et al. \[1986\]](#), [Gay-Balmaz, Marsden, and Ratiu \[2013\]](#).

The present paper grew out of the necessity of a rigorous differential geometric description of some infinite dimensional objects that naturally emerged in the study of dual pairs structures related to fluid dynamics, [Gay-Balmaz and Vizman \[2013\]](#) and [Gay-Balmaz and Vizman \[2014\]](#). Let us mention that a dual pair is an important concept in Poisson geometry. For example, it allows to obtain, in some cases, additional informations about symplectic reduced spaces, see e.g., [Balleier and Wurzbacher \[2012\]](#).

The paper [Gay-Balmaz and Vizman \[2013\]](#) presents a dual pair structure for free boundary fluids. This dual pair is defined on the cotangent bundle $T^*\text{Emb}_{\text{vol}}(S, M)$ of the manifold $\text{Emb}_{\text{vol}}(S, M)$ of all embeddings of a compact volume manifold (S, μ) with boundary ∂S into a volume manifold (M, μ_M) without boundary, both having the same dimension. Its study is based on the existence of a Fréchet differential manifold structure on the infinite dimensional manifolds involved, such as $\text{Emb}_{\text{vol}}(S, M)$, $\text{Emb}_{\text{vol}}(S, M)/\text{Diff}_{\text{vol}}(S)$, $(T\text{Emb}_{\text{vol}}(S, M))/\text{Diff}_{\text{vol}}(S)$.

The paper [Gay-Balmaz and Vizman \[2014\]](#) makes use of the dual pair of momentum maps for ideal fluids found in [Marsden and Weinstein \[1983\]](#) to identify and describe prequantizable coadjoint orbits of the group of Hamiltonian diffeomorphisms. This dual pair is defined on the manifold of embeddings $\text{Emb}(S, M)$, where M is a prequantizable

symplectic manifold. The result is obtained by applying symplectic reduction with respect to the momentum map associated to the action of the group of volume preserving diffeomorphisms $\text{Diff}_{\text{vol}}(S)$, and yields the submanifold of the nonlinear Grassmannian $\text{Gr}^{S,\mu}(M)$ of volume submanifolds of M of type (S, μ) that consists of isotropic submanifolds of M . Here again, the study is based on the existence of a Fréchet differential manifold structure on the principal bundles and nonlinear Grassmannians involved.

Our study extends or complements earlier work on the infinite dimensional manifold structures on principal bundles of embeddings, such as e.g., [Hamilton \[1982\]](#), [Binz and Fischer \[1981\]](#), [Kriegl and Michor \[1997\]](#), [Molitor \[2008\]](#), [Molitor \[2012\]](#).

Below we recall some basic definitions related to calculus on Fréchet manifolds, see e.g. [Hamilton \[1982\]](#). A more general calculus (the convenient calculus) is developed in [Kriegl and Michor \[1997\]](#), to deal with manifold modeled on a more general class of topological vector spaces than Fréchet spaces. In this paper we will only need the Fréchet setting, but since the two approaches coincide in this case, we will use both references.

Review on Fréchet manifolds. Recall that a *Fréchet space* is a topological vector space whose topology arises from a countable collection of seminorms, and is Hausdorff and complete. For example, the space $\Gamma(V)$ of all smooth (i.e., C^∞) sections of a vector bundle $V \rightarrow S$ over a smooth compact finite dimensional manifold S (possibly with smooth boundary) is a Fréchet space. A countable collection of seminorms is obtained by choosing Riemannian metrics and connections on the vector bundles TS and V and defining

$$\|\sigma\|_n := \sum_{j=0}^n \sup_{s \in S} |\nabla^j \sigma(s)|, \quad n = 0, 1, 2, \dots,$$

where ∇^j denotes the j^{th} covariant derivative of a section $\sigma \in \Gamma(V)$. In particular, the space $C^\infty(S, V)$ of smooth maps on S with values in a finite dimensional vector space V is a Fréchet space.

Let E, F be Fréchet spaces and U be an open subset of E . A continuous map $f : U \subset E \rightarrow F$ is *differentiable at $x \in U$ in the direction e* if the limit

$$Df(x) \cdot e := \lim_{h \rightarrow 0} \frac{f(x + he) - f(x)}{h}$$

exists. We say that the map f is *continuously differentiable* (or C^1) if the limit exists for all $x \in U$ and all $e \in E$ and if the map $Df : (U \subset E) \times E \rightarrow F$, $(x, e) \mapsto Df(x) \cdot e$ is continuous (jointly as a function on a subset of the product). Note that if the Fréchet spaces E and F turn out to be Banach spaces, this definition of C^1 map does not recover the usual definition for Banach spaces since the continuity requirement is weaker. The concept of smooth (C^∞) maps between Fréchet space is defined from the C^1 case in the usual iterative way.

A *Fréchet manifold* modeled on the Fréchet space E is a Hausdorff space M covered by charts (U, φ) where $U \subseteq M$ is open and $\varphi : U \rightarrow \varphi(U) \subseteq E$ is a homeomorphism, such that for any two charts (U, φ) and (U', φ') , the coordinate changes

$$\varphi' \circ \varphi^{-1}|_{\varphi(U \cap U')} : \varphi(U \cap U') \rightarrow \varphi'(U \cap U')$$

are smooth maps between Fréchet spaces. An example of Fréchet manifold is provided by the set $C^\infty(S, M)$ of all smooth maps on a compact finite dimensional manifold S (possibly with smooth boundary) into a finite dimensional manifold M without boundary. The tangent space at f is the Fréchet space $T_f C^\infty(S, M) = \Gamma(f^* TM)$ of all smooth sections of the pull-back vector bundle $f^* TM \rightarrow S$.

A subset $N \subseteq M$ is called a *smooth Fréchet submanifold* of the Fréchet manifold M if there exists a closed subspace F of the Fréchet space E and N is covered by charts (U, φ) of M such that $\varphi(U \cap N) = \varphi(U) \cap F$. These kinds of charts are called *submanifold charts*. The submanifold N is called a *splitting submanifold* if the closed subspace $F \subset E$ is complemented, i.e. there exists a subspace $G \subset E$ such that the addition map $F \times G \rightarrow E$ is a topological isomorphism.

We shall make use of the following regular value theorem [Neeb and Wagemann \[2008\]](#) that follows from an implicit function theorem [Glöckner \[2006\]](#). Let $f : M \rightarrow Q$ be a smooth map from a Fréchet manifold M into a Banach manifold Q . Fix $q_0 \in Q$ and suppose that for all $m \in M$, with $f(m) = q_0$, the tangent map $T_m f : T_m M \rightarrow T_{q_0} Q$ is surjective and the subspace $\ker T_m f$ is complemented, i.e., f is a *submersion*. Then $f^{-1}(q_0)$ is a splitting submanifold of M and $T_m(f^{-1}(q_0)) = \ker(T_m f)$. The hypotheses on the tangent map are equivalent to the existence, for all m with $f(m) = q_0$, of a right inverse $\sigma_m : T_{q_0} Q \rightarrow T_m M$ of $T_m f$. Note that if Q is finite dimensional, then $\ker T_m f$ is necessarily complemented.

Let G be a *Fréchet Lie group*, i.e., a Fréchet manifold with a group structure such that the group multiplication and the inversion are smooth maps. A principal G -bundle in the Fréchet category consists of a Fréchet manifold Q , a (right) action $G \times Q \rightarrow Q$, and a submersion $\pi : Q \rightarrow M$, where M is another Fréchet manifold, such that for each $m \in M$, we can find a neighborhood U of m and a diffeomorphism $\Psi : \pi^{-1}(U) \rightarrow G \times U$ such that (i) the action of G on Q corresponds to the action on $G \times U$ on the first factor by right translation, and (ii) the projection π corresponds to the projection of $G \times U$ onto the second factor.

Manifolds and submanifolds with boundary. Recall that a *differentiable manifold S with boundary* is defined by requiring that its charts φ are bijections from a subset U of S to an open subset of the closed half-plane $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$, endowed with the topology induced from \mathbb{R}^n . Thus, the overlap maps are required to be diffeomorphisms between open subsets of \mathbb{R}_+^n . Recall that a map $f : U \rightarrow V$ between open subsets U, V of \mathbb{R}_+^n is of class C^k if for each point $x \in U$ there exist open neighborhoods U_1 of x and V_1 of $f(x)$ in \mathbb{R}^n and a map $f_1 : U_1 \rightarrow V_1$ of class C^k such that $f|_{U \cap U_1} = f_1|_{U \cap U_1}$. We then define $\mathbf{D}^i f(x) := \mathbf{D}^i f_1(x)$, for $x \in U \cap U_1$ and for all $i = 1, \dots, k$. This definition is independent of the choice of f_1 .

We now recall the definition of a submanifold with boundary (see e.g. [Hirsch, 1976, §4](#)). A subset $P \subset \mathbb{R}^n$ is a *submanifold of \mathbb{R}^n with boundary* if each point of P belongs to the domain of a chart $\varphi : W \rightarrow \mathbb{R}^n$ of \mathbb{R}^n such that

$$\varphi(W \cap P) = \varphi(W) \cap (\mathbb{R}_+^k \times \{0\}) \subset \mathbb{R}^n.$$

Let M be a manifold (with or without boundary). A subset $N \subset M$ is a *submanifold of M with boundary* if each point of N belongs to the domain of a chart $\varphi : U \rightarrow \mathbb{R}^n$ of M

such that $\varphi(U \cap N)$ is a submanifold of \mathbb{R}^n with boundary.

Immersions and embeddings. Let S be a manifold with boundary and M a manifold without boundary. A map $f : S \rightarrow M$ is of class C^k if each of its local representations $f_{loc} : U \rightarrow V$ is of class C^k in the sense recalled above, where U is an open subset of \mathbb{R}_+^n and V is an open subset of \mathbb{R}^m . A smooth map $f : S \rightarrow M$ is an *immersion* if its tangent map $T_s f : T_s S \rightarrow T_{f(s)} M$ is injective for all s , where we recall that the tangent space to S at $s \in \partial S$ has the same dimension as the tangent spaces at interior points. A smooth map $f : S \rightarrow M$ is an *embedding* if it is an injective immersion and a homeomorphism onto its image (with the subspace topology). In this case the image $N := f(S)$ is a smooth submanifold of M with boundary and $\partial N = f(\partial S)$. If S is compact then an injective immersion $f : S \rightarrow M$ is necessarily an embedding.

Given a compact manifold S with or without boundary and M a manifold without boundary, such that $\dim S \leq \dim M$, then the set $\text{Emb}(S, M)$ of all embeddings is an open submanifold of the manifold of all smooth maps $C^\infty(S, M)$ (see e.g. [Michor \[1980\]](#) Proposition 5.3). Similarly, the group of diffeomorphisms $\text{Diff}(S)$ is an open submanifold of $C^\infty(S, S)$. The group operations are smooth, so $\text{Diff}(S)$ carries naturally a Lie group structure.

Nonlinear Grassmannians. Let M be a smooth finite dimensional manifold without boundary and let S be a finite dimensional compact manifold possibly with boundary. The *nonlinear Grassmannian of submanifolds of M of type S* is the set $\text{Gr}^S(M)$ defined by

$$\text{Gr}^S(M) := \{N \subset M \mid N \text{ is a submanifold of } M \text{ diffeomorphic to } S\}.$$

Of course, when S has boundary, then N is a submanifold of M with boundary, in the sense recalled above.

Given $N \in \text{Gr}^S(M)$, by definition there exists $f \in \text{Emb}(S, M)$ such that $N = f(S)$. The embedding f is unique up to the composition on the right by a diffeomorphism in $\text{Diff}(S)$. So we get the bijection

$$\text{Emb}(S, M) / \text{Diff}(S) \longleftrightarrow \text{Gr}^S(M), \quad [f] \mapsto f(S).$$

Two other nonlinear Grassmannians will emerge naturally in this paper as bases of principal bundles. One is the nonlinear Grassmannian $\text{Gr}_0^S(M)$ of submanifolds of M diffeomorphic to S , with the same total volume as S . It arises as the quotient space $\text{Emb}_0(S, M) / \text{Diff}(S)$, where $\text{Emb}_0(S, M)$ is the set of all embeddings that preserve the total volume. It also arises as the quotient space $\text{Emb}_{\text{vol}}(S, M) / \text{Diff}_{\text{vol}}(S)$, of volume preserving embeddings by volume preserving diffeomorphisms. Another one is the nonlinear Grassmannian $\text{Gr}^{S, \mu}(M)$ of volume submanifolds of type (S, μ) , where μ is a volume form on S . It arises as the quotient space $\text{Emb}(S, M) / \text{Diff}_{\text{vol}}(S)$.

Plan of the paper. In Section 2, we show that $\text{Emb}(S, M) \rightarrow \text{Gr}^S(M)$ is a smooth Fréchet principal $\text{Diff}(S)$ -bundle, for any compact manifold S , with $\dim S \leq \dim M$, with or without boundary. In particular, this shows that $\text{Emb}(S, M)$ and $\text{Gr}^S(M)$ are Fréchet manifolds. This extends a result already known in the case $\partial S = \emptyset$,

see [Kriegel and Michor \[1997\]](#). The main difficulty of the proof is the construction of manifold charts for $\text{Gr}^S(M)$, when S has boundary. In Section 3, we show that $\text{Emb}_0(S, M) \rightarrow \text{Gr}_0^S(M)$ is a smooth Fréchet principal $\text{Diff}(S)$ -bundle. This is done by using the regular value theorem recalled above, valid in the Fréchet context. In Section 4 we show that $\text{Gr}^{S, \mu}(M)$ is a Fréchet manifold by identifying it with an associated bundle to $\text{Emb}(S, M)$. Using Moser's decomposition of the diffeomorphism group $\text{Diff}(S)$, we then prove that $\text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M)$ is a smooth Fréchet principal $\text{Diff}_{\text{vol}}(S)$ -bundle. Finally, in Section 5, we show that $\text{Emb}_{\text{vol}}(S, M) \rightarrow \text{Gr}_0^S(M)$ is a smooth Fréchet principal $\text{Diff}_{\text{vol}}(S)$ -bundle, by identifying it with a pull-back bundle of $\text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M)$. In particular, this shows that $\text{Emb}_{\text{vol}}(S, M)$ is a Fréchet manifold. This extends a result known for the set of volume preserving embeddings with nowhere vanishing mean curvature, in the particular case $\partial S = \emptyset$, see [Molitor \[2012\]](#).

In this paper, all the finite dimensional manifolds are assumed to be connected and orientable.

Acknowledgements. This work was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0921.

2 Principal $\text{Diff}(S)$ -bundle structure on embeddings

When the manifold S has no boundary, it is known that $\text{Emb}(S, M)$ and $\text{Gr}^S(M)$ are Fréchet manifolds and that

$$\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M), \quad f \mapsto \pi(f) := f(S)$$

is a $\text{Diff}(S)$ -principal Fréchet bundle. In Section §2.2, we shall extend this result to the case when the manifold S has a non empty boundary. Since the proof in that case is considerably more involved, we first sketch below in Section §2.1 the proof in the boundaryless case, following [Kriegel and Michor \[1997\]](#). A slightly different approach to the principal bundle of embeddings is taken in [Hamilton \[1982\]](#) and [Molitor \[2008\]](#).

Note that if we take as structure group the connected component $\text{Diff}_+(S)$ of orientation preserving diffeomorphisms, then the base space is the nonlinear Grassmannian of oriented submanifolds of M of type S .

2.1 The boundaryless case

Assume that S is compact without boundary. We restrict to the case $\dim S < \dim M$, otherwise every embedding would be a diffeomorphism.

Let us fix $f_0 \in \text{Emb}(S, M)$ and $N_0 := f_0(S)$. We endow M with a Riemannian metric g , so that the normal vector bundle $TM|_{N_0}/TN_0 \rightarrow N_0$ can be identified with the orthogonal subbundle $p : TN_0^\perp \subset TM|_{N_0} \rightarrow N_0$. A normal tubular neighborhood $U \subset M$ of N_0 in M can be built with the exponential map associated to g , namely with the diffeomorphism

$$\tau := \exp_{N_0} : V \subset TN_0^\perp \rightarrow U \subset M, \tag{2.1}$$

where V is an open neighborhood of the zero section of $p : TN_0^\perp \rightarrow N_0$. This diffeomorphism coincides with the identity on N_0 .

We denote the set of V -valued sections of the vector bundle TN_0^\perp by

$$\Gamma_V(TN_0^\perp) := \{\sigma \in \Gamma(TN_0^\perp) \mid \sigma(N_0) \subset V\}. \quad (2.2)$$

It is an open subset of the Fréchet vector space $\Gamma(TN_0^\perp)$.

Definition 2.1 *An embedding of S into M built with a normal vector field $\sigma \in \Gamma_V(TN_0^\perp)$ via the tubular neighborhood diffeomorphism τ as*

$$f^\perp := \tau \circ \sigma \circ f_0 : S \rightarrow M$$

*will be called a **normal embedding relative to N_0** .*

Note that since σ is a section, it is an injective immersion and hence an embedding since S is compact. Since τ and $f_0 : S \rightarrow N_0$ are diffeomorphisms, the composition $\tau \circ \sigma \circ f_0 : S \rightarrow M$ is an embedding.

We denote by $\mathcal{U} \subset \text{Gr}^S(M)$ the set of all submanifolds $N \in \text{Gr}^S(M)$ obtained via normal embeddings relative to N_0 , constructed from sections $\sigma \in \Gamma_V(TN_0^\perp)$. Given a submanifold $N \in \mathcal{U}$, there is a unique normal embedding f^\perp with image N . It can be recovered from an arbitrary embedding $f : S \rightarrow M$ with $f(S) = N$, by writing $f^\perp = f \circ \psi_f^{-1}$, where

$$\psi_f = f_0^{-1} \circ p \circ \tau^{-1} \circ f \in \text{Diff}(S). \quad (2.3)$$

The embedding f^\perp does not depend on the choice of the embedding f , because starting with another embedding of S onto $N \subset M$, $f \circ \psi$ with $\psi \in \text{Diff}(S)$, we get the same normal embedding f^\perp since $\psi_{f \circ \psi} = \psi_f \circ \psi$.

Fréchet manifold structure on $\text{Gr}^S(M)$. A chart around N_0 is defined on the subset $\mathcal{U} \subset \text{Gr}^S(M)$ by

$$\chi : \mathcal{U} \rightarrow \Gamma_V(TN_0^\perp), \quad \chi(N) := \tau^{-1} \circ f^\perp \circ f_0^{-1}, \quad (2.4)$$

where f^\perp the unique normal embedding relative to N_0 such that $f^\perp(S) = N$. The inverse reads $\chi^{-1}(\sigma) = (\tau \circ \sigma)(N_0)$.

Consider two submanifolds $N_i, N_j \in \text{Gr}^S(M)$, let \mathcal{U}_i , resp. \mathcal{U}_j , denote the set of all submanifolds of M that are images of normal embeddings relative to N_i , resp. N_j , and let χ_i , resp. χ_j , be the corresponding charts. Then the chart change $\chi_j \circ \chi_i^{-1}$ reads

$$\chi_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset \Gamma(TN_i^\perp) \rightarrow \chi_j(\mathcal{U}_i \cap \mathcal{U}_j) \subset \Gamma(TN_j^\perp), \quad \sigma \mapsto \tau_j^{-1} \circ (\tau_i \circ \sigma \circ f_0^{-1})^{\perp_j} \circ f_0^{-1},$$

where f^{\perp_j} denotes the normal embedding relative to N_j associated to an embedding f . Notice that the embedding $\tau_i \circ \sigma \circ f_0^{-1} \in \text{Emb}(S, M)$, with image $\chi_i^{-1}(\sigma)$, is a normal embedding relative to N_i , but not necessarily relative to N_j .

The fact that the chart changes are smooth maps between Fréchet spaces follows from the smoothness of the composition and inversion maps, see e.g. Corollary 3.13 and Theorem 43.1 in [Kriegl and Michor \[1997\]](#). Indeed, the map $f \in \text{Emb}(S, M) \mapsto \psi_f =$

$f_0^{-1} \circ p \circ \tau^{-1} \circ f \in \text{Diff}(S)$ is smooth, so the map $f \in \text{Emb}(S, M) \mapsto f^\perp = f \circ \psi_f^{-1} \in \text{Emb}(S, M)$ is also smooth, which shows that $\sigma \mapsto \tau_j^{-1} \circ (\tau_i \circ \sigma \circ f_0^{-1})^{\perp_j} \circ f_0^{-1}$ is smooth.

This proves that $\text{Gr}^S(M)$ is a smooth Fréchet manifold. The connected component of an element $N \in \text{Gr}^S(M)$ is modeled on the Fréchet space $\Gamma(TN^\perp)$ of all smooth sections of the vector bundle $TN^\perp \rightarrow N$.

Note that, while a Riemannian metric is involved in the construction of charts, it is not needed to write the tangent space to $\text{Gr}^S(M)$ at N , since we can make the identification $T_N \text{Gr}^S(M) = \Gamma(TM|_N/TN)$.

Principal $\text{Diff}(S)$ -bundle structure on $\text{Emb}(S, M)$. We now make use of the Fréchet manifold structures on $\text{Diff}(S)$ and $\text{Gr}^S(M)$ to show that $\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M)$, $\pi(f) = f(S)$ is a principal $\text{Diff}(S)$ -bundle.

The local trivializations are built over the open sets \mathcal{U} of all submanifolds of M that are images of normal embeddings relative to N_0 . They are given by

$$\Psi : \pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times \text{Diff}(S), \quad \Psi(f) := (f(S), \psi_f),$$

where $\psi_f = p \circ \tau^{-1} \circ f$ as above. The inverse is given by $\Psi^{-1}(N, \psi) = f^\perp \circ \psi$, where f^\perp is the unique normal embedding relative to N_0 such that $f^\perp(S) = N$. Using the equality $(f^\perp)^{-1} = (p_i \circ \tau_i^{-1})|_N$, we obtain that the transition functions are given by

$$\psi_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \rightarrow \text{Diff}(S), \quad \psi_{ij}(N) = (f^{\perp_i})^{-1} \circ f^{\perp_j}.$$

and are smooth maps.

Note that the tangent map to the projection $\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M)$ sends a vector field $v_f \in \Gamma(f^*TM)$ to the orthogonal projection $(v_f \circ f^{-1})^\perp \in \Gamma(TN^\perp)$, where $N = f(S)$ (see Molitor [2008] Corollary 2.3). The vertical space $V_f \text{Emb}(S, M)$ at f thus consists of vector fields $v_f \in \Gamma(f^*TN)$ such that $v_f(s) \in T_{f(s)}N$. As before, a metric is not needed to write the tangent map, since we can write $T_f\pi(v_f) = [v_f \circ f^{-1}] \in \Gamma(TM/TN)$.

2.2 The case $\partial S \neq \emptyset$

In this section we assume that S is compact with boundary and that $\dim S \leq \dim M$. Note that, contrary to the boundaryless case, the situation $\dim S = \dim M$ is interesting.

The construction of manifold charts for the nonlinear Grassmannian $\text{Gr}^S(M)$ when S has a non empty boundary is considerably involved. We thus start with two instructive examples, before showing the result in general.

2.2.1 The principal bundle $\text{Emb}([0, 1], \mathbb{R}^2) \rightarrow \text{Gr}^{[0, 1]}(\mathbb{R}^2)$

As an illustrative example and to introduce our notations, we consider the case $S = [0, 1]$, $M = \mathbb{R}^2$, with $\partial S = \{0, 1\}$. The construction will be made precise in §2.2.3 below.

We consider the embedding $f_0 : [0, 1] \rightarrow \mathbb{R}^2$ given by $f_0(s) = (s, 0)$. Intuitively, elements of $\text{Gr}^{[0, 1]}(\mathbb{R}^2)$ in a neighborhood of $N_0 = [0, 1] \times \{0\}$ can be obtained by first deforming N_0 in a y -direction and then in a x -direction in \mathbb{R}^2 . Therefore a normal embedding associated to $[0, 1]$ should be uniquely determined from two numbers $\sigma^\dagger = (\sigma_0^\dagger, \sigma_1^\dagger) \in$

$(-\varepsilon/2, \varepsilon/2)^2$ giving the x -deformation, and from a function $\sigma \in C^\infty([0, 1], (-\delta, \delta))$ giving the y -deformation.

To σ^\dagger we associate an embedding $\phi_{\sigma^\dagger} : N_0 \rightarrow \mathbb{R}^2$ given by $\phi_{\sigma^\dagger}(x, 0) = ((1 + \sigma_1^\dagger - \sigma_0^\dagger)x + \sigma_0^\dagger, 0)$, sending $N_0 = [0, 1] \times \{0\}$ to $N_1 = [\sigma_0^\dagger, 1 + \sigma_1^\dagger] \times \{0\} \subset (-\varepsilon/2, 1 + \varepsilon/2) \times \{0\}$. From ϕ_{σ^\dagger} we can build the diffeomorphism $H_{\sigma^\dagger} : [0, 1] \times \mathbb{R} \rightarrow [\sigma_0^\dagger, 1 + \sigma_1^\dagger] \times \mathbb{R}$ defined by $H_{\sigma^\dagger}(x, y) := ((1 + \sigma_1^\dagger - \sigma_0^\dagger)x + \sigma_0^\dagger, y)$ associated to x -deformation. To the function σ is naturally associated the embedding, also denoted σ , given by $\sigma(x, 0) = (x, \sigma(x))$ associated to y -deformation. The normal embedding is therefore $f^\perp(s) := H_{\sigma^\dagger}(\sigma(f_0(s))) = ((1 + \sigma_1^\dagger - \sigma_0^\dagger)s + \sigma_0^\dagger, \sigma(s))$. Such embeddings can be used to construct charts for $\text{Gr}^{[0,1]}(\mathbb{R}^2)$, thereby proving that $\text{Gr}^{[0,1]}(\mathbb{R}^2)$ is a Fréchet manifold modeled on the Fréchet vector space $\mathbb{R}^2 \times C^\infty([0, 1], \mathbb{R})$.

2.2.2 The principal bundle $\text{Emb}(\overline{\mathbb{D}}, \mathbb{R}^3) \rightarrow \text{Gr}^{\overline{\mathbb{D}}}(\mathbb{R}^3)$

We now consider the case of the closed unit disk $\overline{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, with $M = \mathbb{R}^3$ and $\partial\overline{\mathbb{D}} = S^1$.

We consider the embedding $f_0 : \overline{\mathbb{D}} \rightarrow \mathbb{R}^3$ given by $f_0(x, y) = (x, y, 0)$. Intuitively, elements of $\text{Gr}^{\overline{\mathbb{D}}}(\mathbb{R}^3)$ in a neighborhood of $N_0 = \overline{\mathbb{D}} \times \{0\}$ can be obtained by first deforming N_0 in a z -direction and then in a (x, y) -direction perpendicular to the boundary. Therefore a normal embedding associated to $\overline{\mathbb{D}}$ should be uniquely determined from two functions $\sigma^\dagger \in C^\infty(S^1, (-\varepsilon/2, \varepsilon/2)^2)$ and $\sigma \in C^\infty(\overline{\mathbb{D}}, (-\delta, \delta))$.

Let us use polar coordinates (r, θ) in the (x, y) plane and fix a smooth function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ which vanishes in a neighborhood of $(-\infty, -1]$, with $\rho(0) = 1$ and bounded derivative: $0 \leq \rho'(t) \leq 2$ for all $t \in \mathbb{R}$. To the function σ^\dagger we associate an embedding $\phi_{\sigma^\dagger} : N_0 \rightarrow \mathbb{R}^3$ given by $\phi_{\sigma^\dagger}(r, \theta, 0) = (r + \sigma^\dagger(\theta)\rho((r-1)/\varepsilon), \theta, 0)$, which deforms the disk N_0 in a (x, y) -direction perpendicular to the boundary. This results in the manifold $N_1 \subset \overline{\mathbb{D}}_\varepsilon \times \{0\}$ with boundary $\partial N_1 = \{(1 + \sigma^\dagger(\theta), \theta, 0) : \theta \in [0, 2\pi]\} \subset S^1 \times (-\varepsilon, \varepsilon) \times \{0\}$. From ϕ_{σ^\dagger} , we build the embedding $H_{\sigma^\dagger} : \overline{\mathbb{D}} \times \mathbb{R} \rightarrow N_1 \times \mathbb{R}$ defined by $H_{\sigma^\dagger}(r, \theta, z) := (\phi_{\sigma^\dagger}(r, \theta), z)$.

To the function σ is naturally associated the embedding, also denoted σ , given by $\sigma(r, \theta, 0) = (r, \theta, \sigma(r, \theta))$ associated to z -deformations. The normal embedding is therefore $f^\perp(r, \theta) := H_{\sigma^\dagger}(\sigma(f_0(r, \theta))) = (\phi_{\sigma^\dagger}(r, \theta), \sigma(r, \theta))$. Such embeddings can be used to construct charts for $\text{Gr}^{\overline{\mathbb{D}}}(\mathbb{R}^3)$, thereby proving that $\text{Gr}^{\overline{\mathbb{D}}}(\mathbb{R}^3)$ is a Fréchet manifold modeled on the Fréchet vector space $C^\infty(S^1, \mathbb{R}) \times C^\infty(\overline{\mathbb{D}}, \mathbb{R})$.

2.2.3 The general case

Given $N_0 = f_0(S)$ a submanifold of M of type S (compact with boundary), we choose a Riemannian metric g on M such that N_0 is a totally geodesic submanifold of M . We consider the normal bundle TN_0^\perp over N_0 and the line bundle

$$T(\partial N_0)^\dagger := TN_0|_{\partial N_0} \cap T(\partial N_0)^\perp$$

over ∂N_0 . We choose open neighborhoods of zero sections

$$V = \{v \in TN_0^\perp : |v| < \delta\} \quad \text{and} \quad V^\dagger = \{v \in T(\partial N_0)^\dagger : |v| < \varepsilon/2\}$$

such that the Riemannian exponential map induces diffeomorphisms

$$\tau : V \subset TN_0^\perp \rightarrow U \subset M, \quad (2.5)$$

and

$$\tau^\dagger : \{v \in T(\partial N_0)^\dagger : |v| < \varepsilon\} \subset T(\partial N_0)^\dagger \rightarrow U^\dagger \subset M. \quad (2.6)$$

Note that U contains N_0 but is not a neighborhood of N_0 in M because N_0 has boundary. Note also that, since N_0 is totally geodesic, $U^\dagger \cap N_0$ is an open set in N_0 , a so called collar neighborhood. This is crucial for the following construction.

We will build in a canonical way normal embeddings relative to N_0 from given sections $\sigma^\dagger \in \Gamma_{V^\dagger}(T(\partial N_0)^\dagger)$ and $\sigma \in \Gamma_V(TN_0^\perp)$, where $\Gamma_{V^\dagger}(T(\partial N_0)^\dagger)$ and $\Gamma_V(TN_0^\perp)$ denote open subsets of the corresponding Fréchet spaces $\Gamma(T(\partial N_0)^\dagger)$ and $\Gamma(TN_0^\perp)$ as in (2.2). This is done in several steps.

Step I. We extend N_0 through its boundary to a boundaryless submanifold N_0^{ext} of M , having the same dimension as N_0 . This is done by adding to N_0 geodesic segments starting at ∂N_0 in orthogonal direction to ∂N_0 . More precisely, $N_0^{ext} = N_0 \cup U^\dagger$, where $U^\dagger = \exp(\{v \in T(\partial N_0)^\dagger : |v| < \varepsilon\})$. Here we make use of the fact that N_0 is totally geodesic. Thus τ^\dagger from (2.6) is a tubular neighborhood of ∂N_0 in N_0^{ext} . We extend now the diffeomorphism τ from (2.5) via the exponential map to a diffeomorphism

$$\tau^{ext} : V^{ext} \subset (TN_0^{ext})^\perp \rightarrow U^{ext} \subset M. \quad (2.7)$$

Step II. The image of the embedding $\tau^\dagger \circ \sigma^\dagger : \partial N_0 \rightarrow U^\dagger \subset N_0^{ext}$ is the boundary ∂N_1 of a submanifold of $N_1 \subset N_0^{ext}$ with $\dim N_0 = \dim N_1$. We will extend this embedding to an embedding $\phi_{\sigma^\dagger} : N_0 \rightarrow N_0^{ext}$ with image N_1 , which modifies 1_{N_0} only in the collar neighborhood $U^\dagger \cap N_0$. This is done with the help of a smooth function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ which vanishes in a neighborhood of $(-\infty, -1]$, with $\rho(0) = 1$ and bounded derivative: $0 \leq \rho'(t) \leq 2$ for all $t \in \mathbb{R}$. The function ρ is chosen such that, for any $a \in (-\varepsilon/2, \varepsilon/2)$, the function $\lambda_a : t \in [-\varepsilon, 0] \mapsto t + a\rho(t/\varepsilon) \in [-\varepsilon, a]$ is a monotonically increasing bijection. Indeed, $\lambda_a(-\varepsilon) = -\varepsilon$, $\lambda_a(0) = a$, and $\lambda'_a(t) = 1 + a/\varepsilon \rho'(t/\varepsilon) > 1 - 1/2\rho'(t/\varepsilon) > 0$.

The line bundle $T(\partial N_0)^\dagger$ is the normal bundle of ∂N_0 in N_0^{ext} . It is a trivial bundle with trivializing isomorphism simply given by $T(\partial N_0)^\dagger \rightarrow \partial N_0 \times \mathbb{R}$, $v_x \mapsto (x, g(v_x, n))$, where n is the unit outward pointing normal vector field to ∂N_0 in N_0^{ext} . Thus we can identify the open neighborhood V^\dagger of ∂N_0 with $\partial N_0 \times (-\varepsilon/2, \varepsilon/2)$ and the section σ^\dagger with a function $\sigma^\dagger : \partial N_0 \rightarrow (-\varepsilon/2, \varepsilon/2)$. Now, with the help of the embedding

$$h : \partial N_0 \times (-\varepsilon, 0] \rightarrow \partial N_0 \times (-\varepsilon, \varepsilon), \quad h(x, t) = (x, t + \sigma^\dagger(x)\rho(t/\varepsilon)),$$

we can build an embedding $\phi_{\sigma^\dagger} : N_0 \rightarrow N_0^{ext}$ with image N_1 by

$$\phi_{\sigma^\dagger}|_{U^\dagger \cap N_0} = \tau^\dagger \circ h \circ (\tau^\dagger)^{-1} \text{ and } \phi_{\sigma^\dagger}|_{N_0 - U^\dagger} = 1|_{N_0 - U^\dagger}.$$

Step III. We use parallel transport along geodesic segments starting at ∂N_0 in orthogonal direction to ∂N_0 to build a bundle isomorphism H_{σ^\dagger} from TN_0^\perp to TN_1^\perp over the embedding $\phi_{\sigma^\dagger} : N_0 \rightarrow N_0^{ext}$. The map H_{σ^\dagger} restricted to the fibers over $N_0 - U^\dagger$ is the identity, while its restriction to the fiber over $y \in U^\dagger \cap N_0$ is the parallel transport along the geodesic segment that connects y and $\phi_{\sigma^\dagger}(y)$. We remark that parallel transport

along geodesics preserves the norm and maps orthogonal vectors into orthogonal vectors. In particular H_{σ^\dagger} is uniquely determined by an embedding of the (unit) sphere bundle $S(TN_0^\perp)$ to the unit sphere bundle $S((TN_0^{ext})^\perp)$, also denoted by H_{σ^\dagger} .

Step IV. The *normal embedding relative to N_0* , associated to the two sections $\sigma^\dagger \in \Gamma_{V^\dagger}(T(\partial N_0)^\dagger)$ and $\sigma \in \Gamma_V(TN_0^\perp)$, is then defined by

$$f^\perp := \tau^{ext} \circ H_{\sigma^\dagger} \circ \sigma \circ f_0 \in \text{Emb}(S, M). \quad (2.8)$$

This means that we first lift $N_0 = f_0(S)$ in a perpendicular direction by using σ and get a manifold whose boundary is exactly sitting above the boundary of N_0 . Then we apply the diffeomorphism H_{σ^\dagger} , which stretches a collar neighborhood of the boundary of the new manifold in a parallel direction to N_0 according to σ^\dagger .

Let us denote by \mathcal{U} the set of all submanifolds obtained by $N = f^\perp(S)$ for normal embeddings relative to N_0 , associated to $\sigma^\dagger \in \Gamma_{V^\dagger}(T(\partial N_0)^\dagger)$ and $\sigma \in \Gamma_V(TN_0^\perp)$. We now show that, given $N \in \mathcal{U}$, there is a unique normal embedding relative to N_0 whose image is N . We first project the boundary of N down to N_0^{ext} using the projection $(TN_0^{ext})^\perp \rightarrow N_0^{ext}$. This yields a hypersurface Σ in N_0^{ext} that is contained in U^\dagger . Because $N \in \mathcal{U}$, Σ determines a unique section $\sigma^\dagger \in \Gamma_{V^\dagger}(T(\partial N_0)^\dagger)$ such that $\tau^\dagger \circ \sigma^\dagger(N_0) = \Sigma$. By construction of H_{σ^\dagger} , the manifold $H_{\sigma^\dagger}^{-1}(N)$ sits exactly above N_0 : its boundary sits exactly above the boundary of N_0 , and H_{σ^\dagger} sends fibers above ∂N_0 to fibers above Σ . Now, there is a unique $\sigma \in \Gamma_V(TN_0^\perp)$ that lifts N_0 to this manifold $H_{\sigma^\dagger}^{-1}(N)$ above N_0 .

Fréchet manifold charts for $\text{Gr}^S(M)$ can now be built in a similar way as in §2. Indeed, a chart around N_0 is defined on the subset $\mathcal{U} \subset \text{Gr}^S(M)$ by

$$\chi : \mathcal{U} \rightarrow \Gamma_{V^\dagger}(T(\partial N_0)^\dagger) \oplus \Gamma_V(TN_0^\perp), \quad \chi(N) := (\sigma^\dagger, \sigma),$$

where (σ^\dagger, σ) are the unique sections such that the corresponding normal embedding in (2.8) verifies $f^\perp(S) = N_0$. The inverse is $\chi^{-1}(\sigma^\dagger, \sigma) = f^\perp(S)$.

To prove that the change of charts $\chi_j \circ \chi_i^{-1}$ are smooth, we decompose it in a number of steps that consist in smooth mappings. The two charts are defined with submanifolds $N_i = f_i(S)$, $N_j = f_j(S)$, and Riemannian metrics g_i, g_j on M that induce diffeomorphisms $\tau_i^{ext}, \tau_j^{ext}, \tau_i^\dagger, \tau_j^\dagger$ via exponential maps. For $(\sigma^\dagger, \sigma) \in \Gamma_{V_i^\dagger}(T(\partial N_i)^\dagger) \oplus \Gamma_{V_i}(TN_i^\perp)$, these steps are

$$\begin{aligned} \sigma^\dagger &\mapsto H_{i, \sigma^\dagger} \in \text{Emb}(S(TN_i^\perp), S((TN_i^{ext})^\perp)), \\ (\sigma^\dagger, \sigma) &\mapsto f^{\perp i}(\sigma^\dagger, \sigma) = \tau_i^{ext} \circ H_{i, \sigma^\dagger} \circ \sigma \circ f_i \in \text{Emb}(S, M), \\ f^{\perp i} &\mapsto E_j(f^{\perp i}) := (\tau_j^{ext})^{-1} \circ f^{\perp i} \circ f_j^{-1} \in \text{Emb}(N_j, (TN_j^{ext})^\perp), \\ (\sigma^\dagger, \sigma) &\mapsto (\sigma_{new}^\dagger, \sigma_{new}) = \left((\tau_j^\dagger)^{-1} \circ p_{N_j^{ext}} \circ E_j|_{\partial N_j}, H_{j, \sigma_{new}^\dagger}^{-1} \circ E_j \right), \end{aligned}$$

where $E_j = E_j(f^{\perp i}(\sigma^\dagger, \sigma))$ and $p_{N_j^{ext}} : (TN_j^{ext})^\perp \rightarrow N_j^{ext}$.

The principal $\text{Diff}(S)$ -bundle structure on $\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M)$ is now constructed as in the case of a manifold S without boundary in §2.1.

We have thus proved the following result.

Theorem 2.2 *Let S be a smooth compact manifold with smooth boundary and let M be a finite dimensional manifold without boundary. Suppose that $\dim S \leq \dim M$. Then the nonlinear Grassmannian $\text{Gr}^S(M)$ is a smooth Fréchet manifold. The connected component of $N \in \text{Gr}^S(M)$ is modeled on the Fréchet vector space $\Gamma(T(\partial N)^\dagger) \oplus \Gamma(TN^\perp)$. Moreover the projection*

$$\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M), \quad f \mapsto f(S) \quad (2.9)$$

is a Fréchet principal $\text{Diff}(S)$ -bundle.

The tangent map. A tangent vector $v_f \in T_f \text{Emb}(S, M) = \Gamma(f^*TM)$ defines a unique section $v_N := v_f \circ f^{-1} \in \Gamma(TM|_N)$ for $N = f(S)$. Using this notation, the tangent map to the projection π reads $T_f \pi(v_f) = (v_N^\dagger, v_N^\perp) \in T_N \text{Gr}^S(M) = \Gamma(T(\partial N)^\dagger) \oplus \Gamma(TN^\perp)$, where v_N^\dagger is the orthogonal projection of $v_N|_{\partial N}$ on the line bundle $T(\partial N)^\dagger \subset TN|_{\partial N}$ orthogonal to the boundary ∂N , and v_N^\perp is the orthogonal projection of $v_N \in TM|_N$ on the orthogonal bundle TN^\perp . Note that by using the metric g , we have the identification $\Gamma(T(\partial N)^\dagger) = C^\infty(\partial N, \mathbb{R})$ and the first term can be written as $g(v_N|_{\partial N}, n)$, where n is the unit normal outward pointing vector field to ∂N .

The first component vanishes when S has no boundary, as seen in §2.1, while the second component vanishes when S and M have the same dimension, as we will see in more details in the next paragraph.

Note also that the vertical subspace of $T_f \text{Emb}(S, M)$ reads

$$V_f \text{Emb}(S, M) = \{Tf \circ u : u \in \mathfrak{X}_\parallel(S)\} = \{v \circ f : v \in \mathfrak{X}_\parallel(f(S))\},$$

where $\mathfrak{X}_\parallel(S)$ denotes the space of smooth vector fields on S parallel to the boundary. In the equality above, we defined $v := f_* u$. Since u is an arbitrary vector field on S parallel to the boundary, v is an arbitrary vector field on $f(S)$ parallel to the boundary.

We consistently have an isomorphism $T_f \text{Emb}(S, M)/V_f \text{Emb}(S, M) \simeq T_N \text{Gr}^S(M)$, for $N = f(S)$.

A metric is not needed to identify the tangent space, since we can make the identification $T_N \text{Gr}^S(M) = \Gamma(TN|_{\partial N}/T\partial N) \times \Gamma(TM|_N/TN)$, in which case we can write $T_f \pi(v_f) = ([v_f \circ f^{-1}|_{\partial N}], [v_f \circ f^{-1}])$.

Embeddings of manifolds of the same dimension. In the special case $\dim S = \dim M$ parts of the construction of a normal embedding relative to N_0 simplify because the normal bundle over N_0 is zero. The normal embedding is associated to a single section, namely the section σ^\dagger of the bundle $T(\partial N_0)^\dagger = T(\partial N_0)^\perp$ over ∂N_0 . The Riemannian metric g on M can be arbitrarily chosen and the manifold N_0^{ext} that extends N_0 is just an open subset of M . The Fréchet manifold $\text{Gr}^S(M)$ is modeled on $\Gamma(T(\partial N_0)^\dagger) = \Gamma(T(\partial N_0)^\perp)$. Rewritten in this case, Theorem 2.2 reads as follows.

Proposition 2.3 *Let S be a smooth compact manifold with smooth boundary and let M be a finite dimensional manifold without boundary. Suppose that $\dim S = \dim M$. Then the nonlinear Grassmannian $\text{Gr}^S(M)$ is a smooth Fréchet manifold. The connected component of $N \in \text{Gr}^S(M)$ is modeled on the Fréchet vector space $\Gamma(T(\partial N)^\perp) \simeq C^\infty(\partial N)$.*

Moreover the projection

$$\pi : \text{Emb}(S, M) \rightarrow \text{Gr}^S(M), \quad f \mapsto f(S) \quad (2.10)$$

is a Fréchet principal $\text{Diff}(S)$ -bundle.

Note that the tangent map to π read $T_f\pi(v_f) = v_N^\dagger \in \Gamma(T(\partial N)^\perp)$, where $v_N := v_f \circ f^{-1}$ and v_N^\dagger is the orthogonal projection of $v_N|_{\partial N}$ to the line bundle $T(\partial N)^\perp$. It can also be written as $T_f\pi(v_f) = g(v_N|_{\partial N}, n) \in C^\infty(\partial N)$, where g is the Riemannian metric on M and n is the unit normal outward pointing vector field to ∂N .

3 Principal $\text{Diff}(S)$ -bundle structure on embeddings that preserve the total volume

In this section we consider the subset $\text{Emb}_0(S, M) \subset \text{Emb}(S, M)$ of all embeddings $f : S \rightarrow M$ that preserve the total volume. This set arises as the total space of a $\text{Diff}(S)$ -principal bundle over the nonlinear Grassmannian $\text{Gr}_0^S(M)$ of submanifolds N diffeomorphic to S , with same volume with S .

We first consider below the case $\dim S = \dim M$ since it is considerably simpler. For the case $\dim S < \dim M$ a Riemannian metric is needed on M .

3.1 The case $\dim S = \dim M$

We fix volume forms μ on S and μ_M on M .

Proposition 3.1 *The set of embeddings that preserve the total volume*

$$\text{Emb}_0(S, M) := \left\{ f \in \text{Emb}(S, M) : \int_{f(S)} \mu_M = \int_S \mu \right\}$$

is a splitting submanifold of $\text{Emb}(S, M)$, with tangent space

$$T_f \text{Emb}_0(S, M) = \left\{ v_f \in T_f \text{Emb}(S, M) : \int_{\partial N} i_{\partial N}^* \mathbf{i}_{v_f \circ f^{-1}} \mu_M = 0 \right\}.$$

Proof. Consider the smooth map $\text{vol} : \text{Emb}(S, M) \rightarrow \mathbb{R}$, $\text{vol}(f) := \int_S f^* \mu_M$ with derivative

$$\mathbf{d} \text{vol}(f) \cdot v_f = \int_S \mathbf{d}f^* \mathbf{i}_{v_f \circ f^{-1}} \mu_M = \int_{\partial S} i_{\partial S}^* f^* \mathbf{i}_{v_f \circ f^{-1}} \mu_M = \int_{\partial N} i_{\partial N}^* \mathbf{i}_{v_f \circ f^{-1}} \mu_M,$$

where $i_{\partial S} : \partial S \rightarrow S$ is the inclusion. This map is a submersion with values in a finite dimensional manifold, so the regular value theorem [Neeb and Wagemann \[2008\]](#) mentioned in the Introduction can be applied to the submersion vol to show that the set of embeddings that preserve the total volume $\text{Emb}_0(S, M) = \text{vol}^{-1}(\int_S \mu)$ is a submanifold of $\text{Emb}(S, M)$ of codimension one. ■

Note that the kernel can be equivalently written

$$\ker(\mathbf{d} \operatorname{vol}(f)) = \{v_f \in T_f \operatorname{Emb}(S, M) : i_{\partial S}^* f^* \mathbf{i}_{v_f \circ f^{-1}} \mu_M \in \Omega_{exact}^{k-1}(\partial S)\},$$

where $k = \dim S = \dim M$. When the volume form μ_M is associated to a Riemannian metric g on M , then we can write $\mathbf{i}_{v_f \circ f^{-1}} \mu_M = g(v_f \circ f^{-1}, n) \mu_{\partial N}$, where n is the unit outward pointing normal vector field to ∂N and $\mu_{\partial N}$ is the volume form induced on the boundary. In this case, the tangent space $T_f \operatorname{Emb}_0(S, M)$ consists of vector fields $v_f \in \Gamma(f^* TM)$ such that

$$\int_{\partial N} g(v_f \circ f^{-1}|_{\partial N}, n) \mu_{\partial N} = 0.$$

In the same way, using the submersion $\operatorname{vol} : \operatorname{Gr}^S(M) \rightarrow \mathbb{R}$, $\operatorname{vol}(N) = \int_N \mu_M$, we obtain that the nonlinear Grassmannian of submanifolds of M of same volume as S

$$\operatorname{Gr}_0^S(M) := \left\{ N \subset M : N \text{ submanif. diffeom. to } S, \int_N \mu_M = \int_S \mu \right\} \quad (3.1)$$

is a codimension one submanifold of $\operatorname{Gr}^S(M)$, with tangent space

$$T_N \operatorname{Gr}_0^S(M) = \left\{ w_N \in \Gamma(T(\partial N)^\perp) : \int_{\partial N} i_{\partial N}^* \mathbf{i}_{w_N} \mu_M = 0 \right\}.$$

Indeed, the derivative is

$$\mathbf{d} \operatorname{vol}(N) \cdot w_N = \int_N \mathbf{d} i_N^* \mathbf{i}_{w_N} \mu_M = \int_{\partial N} i_{\partial N}^* \mathbf{i}_{w_N} \mu_M,$$

for all $w_N \in \Gamma(TN^\perp)$.

As above, when μ_M is associated to a Riemannian metric g on M , a section $w_N \in \Gamma(T(\partial N)^\perp)$ belongs to the tangent space $T_N \operatorname{Gr}_0^S(M)$ if and only if $\int_N g(w_N, n) \mu_{\partial N} = 0$. The tangent space can thus be identified with $C_0^\infty(\partial N) = \{h \in C^\infty(N, \mathbb{R}) \mid \int_N h \mu_{\partial M} = 0\}$, where $h = g(w_N, n)$, i.e., $w_N = hn$.

We thus have proved the following theorem.

Proposition 3.2 *The projection*

$$\pi_0 : \operatorname{Emb}_0(S, M) \rightarrow \operatorname{Gr}_0^S(M), \quad f \mapsto f(S)$$

is a Fréchet principal $\operatorname{Diff}(S)$ -bundle.

The projection π_0 is the restriction of the projection π in (2.10), so the expression of its tangent map is the restriction of that of π .

3.2 The case $\dim S < \dim M$

We fix a volume form μ on S and a Riemannian metric g on M with induced volume form $\mu(g)$. This metric induces on any submanifold N of M a volume form $\mu(g_N)$, where g_N is the Riemannian metric induced on N . We denote by $H_N \in \Gamma(TN^\perp)$ the trace of the second fundamental form $\Pi_N : TN \times TN \rightarrow TN^\perp$ of the submanifold N , i.e. the mean curvature vector field.

Proposition 3.3 *The set of embeddings that preserve the total volume*

$$\text{Emb}_0(S, M) := \left\{ f \in \text{Emb}(S, M) : \int_{f(S)} \mu(g_{f(S)}) = \int_S \mu \right\}$$

is a splitting submanifold of $\text{Emb}(S, M)$, whose tangent space at f is

$$\left\{ v_f \in T_f \text{Emb}(S, M) : \int_{\partial N} g(v_f \circ f^{-1}, n) \mu(g_{\partial N}) = \int_N g(H_N, v_f \circ f^{-1}) \mu(g_N) \right\}.$$

Proof. Consider the smooth map $\text{vol} : \text{Emb}(S, M) \rightarrow \mathbb{R}$, $\text{vol}(f) := \int_S f^* \mu(g_{f(S)})$ with derivative

$$\mathbf{d} \text{vol}(f) \cdot v_f = \int_{\partial S} i_{\partial S}^* f^* \mathbf{i}_{(v_f \circ f^{-1})^\top} \mu(g_{f(S)}) - \int_S g(H_{f(S)} \circ f, v_f) f^* \mu(g_{f(S)}),$$

see e.g. [Gallot, Hulin, and Lafontaine \[2004\]](#) (the proof of Theorem 5.20). This is a submersion with values in a finite dimensional manifold for which we can apply the regular value theorem [Neeb and Wagemann \[2008\]](#) mentioned in the Introduction to show that the set of embeddings that preserve the total volume $\text{Emb}_0(S, M) = \text{vol}^{-1}(\int_S \mu)$ is a submanifold of $\text{Emb}(S, M)$ of codimension one. ■

In the same way, using the submersion $\text{vol} : \text{Gr}^S(M) \rightarrow \mathbb{R}$, $\text{vol}(N) = \int_N \mu(g_N)$, with derivative

$$\mathbf{d} \text{vol}(N) \cdot (w_{\partial N}, w_N) = \int_{\partial N} g(w_{\partial N}, n) \mu(g_{\partial N}) - \int_N g(H_N, w_N) \mu(g_N), \quad (3.2)$$

for $(w_{\partial N}, w_N) \in T_N \text{Gr}^S(M) = \Gamma(T(\partial N)^\dagger) \oplus \Gamma(TN^\perp)$, we obtain that the nonlinear Grassmannian

$$\text{Gr}_0^S(M) := \left\{ N \subset M : N \text{ submanif. diffeom. to } S, \int_N \mu(g_N) = \int_S \mu \right\} \quad (3.3)$$

of all submanifolds of M of same volume as S , is a codimension one submanifold of $\text{Gr}^S(M)$. From (3.2), the tangent space to $\text{Gr}_0^S(M)$ reads

$$T_N \text{Gr}_0^S(M) = \left\{ (w_{\partial N}, w_N) : \int_{\partial N} g(w_{\partial N}, n) \mu(g_{\partial N}) = \int_N g(H_N, w_N) \mu(g_N) \right\}. \quad (3.4)$$

As in the previous case we get the following result.

Proposition 3.4 *The projection*

$$\pi_0 : \text{Emb}_0(S, M) \rightarrow \text{Gr}_0^S(M), \quad f \mapsto f(S)$$

is a Fréchet principal $\text{Diff}(S)$ -bundle.

The projection π_0 is the restriction of the projection π in (2.9), so the expression of its tangent map is the restriction of that of π , namely $T_f \pi_0(v_f) = (v_N^\dagger, v_N^\top) = (w_{\partial N}, w_N) \in T_N \text{Gr}_0^S(M)$, see (3.4), where $v_N := v_f \circ f^{-1}$.

Note that $V_f \text{Emb}_0(S, M) = V_f \text{Emb}(S, M)$, for all $f \in \text{Emb}_0(S, M)$, and that we consistently have $T_f \text{Emb}_0(S, M)/V_f \text{Emb}_0(S, M) \simeq T_N \text{Gr}_0^S(M)$.

4 Principal $\text{Diff}_{\text{vol}}(S)$ -bundle structure on embeddings

In this section we consider another principal bundle structure on $\text{Emb}(S, M)$, that naturally arises in Gay-Balmaz and Vizman [2014] in the context of symplectic reduction applied to the dual pair of momentum maps for the ideal fluid. Here we assume that S is compact, possibly with boundary, and $\dim S \leq \dim M$.

Nonlinear Grassmannian of volume submanifolds. Let us assume that S is endowed with a volume form μ . This fixes an orientation on S that we use whenever we integrate over S . The *nonlinear Grassmannian of volume submanifolds of type (S, μ)* is

$$\text{Gr}^{S, \mu}(M) := \left\{ (N, \nu) : N \in \text{Gr}^S(M), \nu \in \text{Vol}(N), \int_N \nu = \int_S \mu \right\}, \quad (4.1)$$

where the orientation on N is the one induced by the volume form ν .

We consider the set of volume forms

$$\text{Vol}_\mu(S) = \left\{ \rho \in \text{Vol}(S) : \int_S \rho = \pm \int_S \mu \right\}, \quad (4.2)$$

union of two convex connected components. The fiber of the forgetting map

$$\pi_1 : \text{Gr}^{S, \mu}(M) \rightarrow \text{Gr}^S(M), \quad (N, \nu) \mapsto \pi_1(N, \nu) = N \quad (4.3)$$

is isomorphic to $\text{Vol}_\mu(S)$. Indeed, the fiber of $\text{Gr}^{S, \mu}(M)$ above N is $\{\nu \in \text{Vol}(N) : \int_N \nu = \int_S \mu\}$, with the orientation on N induced from ν . This means that if ν belongs to the fiber, then $-\nu$ is in the fiber too, so that the fiber is isomorphic to $\text{Vol}_\mu(S)$.

Next we show that $\text{Gr}^{S, \mu}(M)$ can be identified with the associated bundle to the principal $\text{Diff}(S)$ -bundle $\text{Emb}(S, M) \rightarrow \text{Gr}^S(M)$ with respect to the natural action of the diffeomorphism group on $\text{Vol}_\mu(S)$. This is made precise in the following proposition.

Proposition 4.1 *The nonlinear Grassmannian of volume submanifolds of type (S, μ) with the forgetting map $\pi_1 : \text{Gr}^{S, \mu}(M) \rightarrow \text{Gr}^S(M)$ is a Fréchet fiber bundle with fiber $\text{Vol}_\mu(S)$, that can be identified with the associated bundle*

$$\text{Emb}(S, M) \times_{\text{Diff}(S)} \text{Vol}_\mu(S) \rightarrow \text{Gr}^S(M), \quad [f, \rho] \mapsto f(S) = N.$$

In particular, $\text{Gr}^{S,\mu}(M)$ is a Fréchet manifold. The connected component of $(N, \nu) \in \text{Gr}^{S,\mu}(M)$ is modeled on the Fréchet vector space $\Gamma(T(\partial N)^\dagger) \oplus \Gamma(TN^\perp) \oplus \Omega_0^k(N)$, where

$$\Omega_0^k(N) := \left\{ \sigma \in \Omega^k(N) : \int_N \sigma = 0 \right\}, \quad k = \dim S. \quad (4.4)$$

Proof. The fiber bundle isomorphism is given by

$$(N, \nu) \in \text{Gr}^{S,\mu}(M) \mapsto [f, f^*\nu] \in \text{Emb}(S, M) \times_{\text{Diff}(S)} \text{Vol}_\mu(S), \quad (4.5)$$

where f is any embedding with image N . We first check that $f^*\nu \in \text{Vol}_\mu(S)$. Indeed, $\int_S f^*\nu = \pm \int_{f(S)} \nu$, since the orientation on $f(S) = N$ is induced by the volume form ν , which might or might not coincide with that induced by $f_*\mu$ (recall that the integral over S uses the orientation induced by μ). But $\int_N \nu = \int_S \mu$, so that $\int_S f^*\nu = \pm \int_S \mu$. Then we verify that the map is well defined, i.e. the result doesn't depend on the choice of the embedding with image N : $[f \circ \varphi, (f \circ \varphi)^*\nu] = [f \circ \varphi, \varphi^* f^*\nu] = [f, f^*\nu]$ for any $\varphi \in \text{Diff}(S)$.

The inverse of (4.5) is given by

$$[f, \rho] \in \text{Emb}(S, M) \times_{\text{Diff}(S)} \text{Vol}_\mu(S) \mapsto (f(S), f_*\rho) \in \text{Gr}^{S,\mu}(M).$$

We have $0 < \int_{f(S)} f_*\rho = \pm \int_S \rho$ with orientation on $f(S)$ induced by the volume form $f_*\rho$, which might or might not coincide with that induced by $f_*\mu$. But $0 < \int_S \mu = \pm \int_S \rho$, so $\int_{f(S)} f_*\rho = \int_S \mu$, which shows that $(f(S), f_*\rho) \in \text{Gr}^{S,\mu}(M)$. It remains to verify that the map doesn't depend on the choice of the representative in the associated bundle: $[f \circ \varphi, \varphi^*\rho] \mapsto ((f \circ \varphi)(S), (f \circ \varphi)_*\varphi^*\rho) = (f(S), f_*\rho)$ for any $\varphi \in \text{Diff}(S)$.

Using the principal bundle charts for $\text{Emb}(S, M)$, it is now standard to build bundle manifold charts for the associated bundle. Indeed, we have the isomorphisms

$$\pi_1^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{U}) \times_{\text{Diff}(S)} \text{Vol}_\mu(S) \cong (\mathcal{U} \times \text{Diff}(S)) \times_{\text{Diff}(S)} \text{Vol}_\mu(S) \cong \mathcal{U} \times \text{Vol}_\mu(S).$$

This shows that $\pi_1 : \text{Gr}^{S,\mu}(M) \rightarrow \text{Gr}^S(M)$ is a smooth Fréchet fiber bundle. Since $\text{Gr}^{S,\mu}(M)$ is an associated bundle, its model Fréchet space is isomorphic to the sum of the model Fréchet space of $\text{Gr}^S(M)$ and the model Fréchet space of $\text{Vol}_\mu(S)$, i.e., $\Gamma(T(\partial N)^\dagger) \oplus \Gamma(TN^\perp)$ and $\Omega_0^k(N)$. ■

Decomposition of the group of diffeomorphism. A theorem of Moser ensures, for a given volume form μ on S , the existence of a smooth map $B : \text{Vol}_\mu(S) \rightarrow \text{Diff}(S)$ defined on the space (4.2) of volume forms on S with total volume $\int_S \mu$, such that $B(\nu)_*\mu = \nu$.

Using this result one shows that the group of diffeomorphisms of S splits smoothly as the product of the group of volume preserving diffeomorphisms and $\text{Vol}_\mu(S)$:

$$\text{Diff}(S) = \text{Diff}_{\text{vol}}(S) \times \text{Vol}_\mu(S). \quad (4.6)$$

This result is valid also for manifolds with smooth boundary, see Theorem 5.1 and 8.6 in Ebin and Marsden [1970]. Recall that the diffeomorphism (4.6) reads $\varphi \mapsto (\varphi_{\text{vol}}, \varphi_*\mu) = (B(\varphi_*\mu)^{-1} \circ \varphi, \varphi_*\mu)$ with inverse $(\psi, \nu) \mapsto B(\nu) \circ \psi$.

The version of (4.6) that appears in Ebin and Marsden [1970] is $\text{Diff}_+(S) = \text{Diff}_{\text{vol}}(S) \times \text{Vol}_\mu^+(S)$, where $\text{Diff}_+(S)$ denotes the group of orientation preserving diffeomorphisms and $\text{Vol}_\mu^+(S) = \{\rho \in \text{Vol}(S) : \int_S \rho = \int_S \mu\}$.

Therefore, every diffeomorphism $\varphi \in \text{Diff}(S)$ can be transformed into a volume preserving diffeomorphism

$$\varphi_{\text{vol}} := B(\varphi_*\mu)^{-1} \circ \varphi \in \text{Diff}_{\text{vol}}(S). \quad (4.7)$$

There is another possibility to obtain a volume preserving diffeomorphism out of an ordinary diffeomorphism φ , namely, by composing on the right with $B(\varphi^*\mu)$.

Principal $\text{Diff}_{\text{vol}}(S)$ -bundle structure on $\text{Emb}(S, M)$. In the same way as $\pi : f \in \text{Emb}(S, M) \mapsto f(S) \in \text{Gr}^S(M)$ is a principal $\text{Diff}(S)$ -bundle, we now show that the map

$$\pi_{\text{vol}} : \text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M), \quad f \mapsto \pi_{\text{vol}}(f) = (f(S), f_*\mu), \quad (4.8)$$

is a principal $\text{Diff}_{\text{vol}}(S)$ -bundle.

In order to prove that π_{vol} is surjective, given an arbitrary $(N, \nu) \in \text{Gr}^{S, \mu}(M)$, we need to find $f \in \text{Emb}(S, M)$ such that $f(S) = N$ and $f_*\mu = \nu$. We know that there exists $f' \in \text{Emb}(S, M)$ such that $N = f'(S)$. Let us consider $(f')^*\nu \in \text{Vol}(S)$. Since $\int_S \mu = \int_N \nu = \int_S (f')^*\nu$, there exists $\varphi = B((f')^*\nu) \in \text{Diff}(S)$ such that $\varphi_*((f')^*\nu) = \mu$ (by Moser's theorem). Then the requested embedding is $f := f' \circ \varphi^{-1}$. On the other hand, if the embeddings f_1 and f_2 satisfy $\pi_{\text{vol}}(f_1) = \pi_{\text{vol}}(f_2)$, then there is a diffeomorphism φ of S such that $f_2 = f_1 \circ \varphi$, while the identity $(f_1)_*\mu = (f_2)_*\mu$ ensures that the diffeomorphism φ is volume preserving. For the moment we get a bijection:

$$\text{Emb}(S, M) / \text{Diff}_{\text{vol}}(S) \longleftrightarrow \text{Gr}^{S, \mu}(M), \quad [f] \mapsto (f(S), f_*\mu).$$

Local trivializations. They can be built over the open sets $\mathcal{V} = \pi_1^{-1}(\mathcal{U}) \subset \text{Gr}^{S, \mu}(M)$, preimages by the forgetting map (4.3) of the open sets \mathcal{U} of all submanifolds of M that are images of normal embeddings. We take

$$\Psi : \pi_{\text{vol}}^{-1}(\mathcal{V}) \rightarrow \mathcal{V} \times \text{Diff}_{\text{vol}}(S), \quad \Psi(f) = (\pi_{\text{vol}}(f), (\psi_f)_{\text{vol}}),$$

where $(\psi_f)_{\text{vol}} := B(\psi_{f*}\mu)^{-1} \circ \psi_f$ with ψ_f from (2.3). Its inverse reads

$$\Psi^{-1}((N, \nu), \varphi) = f^\perp \circ B((f^\perp)^*\nu) \circ \varphi,$$

where f^\perp is the unique normal embedding such that $f^\perp(S) = N$. Indeed,

$$\begin{aligned} \Psi(f^\perp \circ B((f^\perp)^*\nu) \circ \varphi) &= ((f^\perp(S), f_*^\perp B((f^\perp)^*\nu) \varphi_* \mu), (B((f^\perp)^*\nu) \circ \varphi)_{\text{vol}}) \\ &= ((f(S), f_*^\perp B((f^\perp)^*\nu) \mu), B(B((f^\perp)^*\nu)_* \varphi_* \mu)^{-1} \circ B((f^\perp)^*\nu) \circ \varphi) \\ &= ((f(S), \nu), \varphi) \end{aligned}$$

and

$$\Psi^{-1}(\pi_{\text{vol}}(f), (\psi_f)_{\text{vol}}) = f^\perp \circ B((f^\perp)^*\nu) \circ B((\psi_f)_*\mu)^{-1} \circ \psi_f = f^\perp \circ \psi_f = f.$$

The transition functions are smooth since they express as:

$$\phi_{ij} : \mathcal{V}_i \cap \mathcal{V}_j \rightarrow \text{Diff}_{\text{vol}}(S), \quad \phi_{ij}(N, \nu) = (f^{\perp_i})^{-1} \circ f^{\perp_j}.$$

We have thus proved the following result.

Proposition 4.2 *Let S be a compact manifold, possibly with boundary, and let M be a finite dimensional manifold without boundary. Let μ be a volume form on S . Then, the projection (4.8)*

$$\pi_{\text{vol}} : \text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M)$$

is a Fréchet principal $\text{Diff}_{\text{vol}}(S)$ -bundle.

Note that the vertical subspace of $T_f \text{Emb}(S, M)$, relative to the $\text{Diff}_{\text{vol}}(S)$ -principal bundle structure, reads

$$V_f \text{Emb}(S, M) = \{Tf \circ u : u \in \mathfrak{X}_{\parallel}(S, \mu)\} = \{v \circ f : v \in \mathfrak{X}_{\parallel}(f(S), f_*\mu)\}, \quad (4.9)$$

where $\mathfrak{X}_{\parallel}(S, \mu)$ denotes the space of smooth vector fields on S parallel to the boundary and divergence-free relative to μ . In the equality above, we defined $v := f_*u$. Since u is an arbitrary vector field on S parallel to the boundary and divergence-free relative to μ , v is an arbitrary vector field on $f(S)$ parallel to the boundary and divergence-free relative to the volume for $f_*\mu$ on $f(S)$.

Let $\Omega_n^{k-1}(N) = \{\alpha \in \Omega^{k-1}(N) : i_{\partial N}^* \alpha = 0\}$ denote the space of differential $(k-1)$ -forms normal to the boundary. Then the space of $\Omega_0^k(N)$ of k -forms with zero integral, see (4.4), can be written as $\Omega_0^k(N) = \mathbf{d}\Omega_n^{k-1}(N)$ Ebin and Marsden [1970]. Therefore, the tangent space $T_{(N, \nu)} \text{Gr}^{S, \mu}(M)$ can be identified with the product space $\Gamma(T(\partial N)^\dagger) \times \Gamma(TN^\perp) \times \mathbf{d}\Omega_n^{k-1}(N)$, for any $(N, \nu) \in \text{Gr}^{S, \mu}(M)$, by Proposition 4.1.

The next proposition gives another expression for the tangent space $T_{(N, \nu)} \text{Gr}^{S, \mu}(M)$, which permits to express the tangent map $T_f \pi_{\text{vol}}$ in a simple way.

Proposition 4.3 *The tangent space to the nonlinear volume Grassmannian can be identified with*

$$T_{(N, \nu)} \text{Gr}^{S, \mu}(M) = E_{(N, \nu)} \times \Gamma(TN^\perp) \subset \Gamma(T(\partial N)^\dagger) \times \mathbf{d}\Omega^{k-1}(N) \times \Gamma(TN^\perp),$$

where

$$E_{(N, \nu)} := \{(w_{\partial N}, \mathbf{d}\alpha) \in \Gamma(T(\partial N)^\dagger) \times \mathbf{d}\Omega^{k-1}(N) : w_{\partial N} \nu_{\partial} = i_{\partial N}^* \alpha\}.$$

Let g be a Riemannian metric on M . Given $f \in \text{Emb}(S, M)$ such that $f(S) = N$ and $f_*\mu = \nu$, the tangent map $T_f \pi_{\text{vol}}$ becomes

$$T_f \pi_{\text{vol}} : T_f \text{Emb}(S, M) \rightarrow T_{(N, \nu)} \text{Gr}^{S, \mu}(M), \quad T_f \pi_{\text{vol}}(v_f) = \left(v_N^\dagger, \mathcal{L}_{v_N^\top} \nu, v_N^\perp \right).$$

Here $v_N := v_f \circ f^{-1} \in \Gamma(TM|_N)$ and $v_N = v_N^\perp + v_N^\top$ denotes the orthogonal decomposition along $N \subset M$, while $v_N^\dagger \in \Gamma(T(\partial N)^\dagger)$ denotes the orthogonal projection of $v_N^\top|_{\partial N}$ to the line bundle $T(\partial N)^\dagger$.

Proof. Since $\pi_{\text{vol}} : \text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M)$ is a principal $\text{Diff}_{\text{vol}}(S)$ -bundle, we have the identification

$$T_{(N, \nu)} \text{Gr}^{S, \mu}(M) = T_f \text{Emb}(S, M) / V_f \text{Emb}(S, M),$$

where $V_f \text{Emb}(S, M)$ is the vertical space (4.9) at $f \in \text{Emb}(S, M)$. Let $N = f(S)$ and $\nu = f_*\mu$. The map

$$v_f \in T_f \text{Emb}(S, M) \longmapsto \left(v_N^\dagger, \mathcal{L}_{v_N^\top} \nu, v_N^\perp \right) \in E_{(N, \nu)} \times \Gamma(TN^\perp) \quad (4.10)$$

is well defined because $\mathcal{L}_{v_N^\top} \nu = \mathbf{d}(\mathbf{i}_{v_N^\top} \nu)$ and the formula

$$i_{\partial N}^*(\mathbf{i}_{v_N^\top} \nu) = g(v_N^\top, n) \nu_\partial = v_N^\dagger \nu_\partial, \quad (4.11)$$

which shows that $\left(v_N^\dagger, \mathcal{L}_{v_N^\top} \nu \right) \in E_{(N, \nu)}$.

We now show it is a linear surjective map with kernel $V_f \text{Emb}(S, M)$. Let $v_f \in \Gamma(f^*TM)$ be in the kernel of (4.10), so that $v_N^\dagger = 0$, $v_N^\perp = 0$ and $\mathcal{L}_{v_N^\top} \nu = 0$. This means that $v_N = v_N^\top \in \mathfrak{X}_\parallel(N, \nu)$. Its pull-back by the diffeomorphism $f : S \rightarrow f(S) = N$, denoted by u , belongs to $\mathfrak{X}_\parallel(S, \mu)$. Hence $v_f = Tf \circ u \in V_f \text{Emb}(S, M)$.

For the surjectivity we consider a triple $(w_{\partial N}, \mathbf{d}\alpha, w_N^\perp) \in E_{(N, \nu)} \times \Gamma(TN^\perp)$, so there is a relation between the first two components $w_{\partial N} \nu_\partial = i_{\partial N}^* \alpha$. The differential form $\alpha \in \Omega^{k-1}(N)$ determines the vector field $w_N^\top \in \mathfrak{X}(N)$ by $\mathbf{i}_{w_N^\top} \nu = \alpha$. Now we can define $v_f := v_N \circ f$, where $v_N := w_N^\perp + w_N^\top$, because of the formula (4.11). ■

Remark 4.4 The two descriptions of the tangent space $T_{(N, \nu)} \text{Gr}^{S, \mu}(M)$ are of course isomorphic. This can be seen directly as follows. Using tubular neighborhoods as in Section 2.2.3, we assign to each $w_{\partial N} \in \Gamma(T(\partial N)^\dagger)$ a differential form $\beta(w_{\partial N}) \in \Omega^{k-1}(N)$ that extends $w_{\partial N} \nu_\partial \in \Omega^{k-1}(\partial N)$, i.e., $w_{\partial N} \nu_\partial = i_{\partial N}^* \beta(w_{\partial N})$. The isomorphism reads $(w_{\partial N}, \mathbf{d}\alpha) \in E_{(N, \nu)} \mapsto (w_{\partial N}, \mathbf{d}(\alpha - \beta(w_{\partial N}))) \in \Gamma(T(\partial N)^\dagger) \times \Omega_0^k(N)$ with inverse $(w_{\partial N}, \mathbf{d}\lambda) \in \Gamma(T(\partial N)^\dagger) \times \Omega_0^k(N) \mapsto (w_{\partial N}, \mathbf{d}(\lambda + \beta(w_{\partial N}))) \in E_{(N, \nu)}$.

5 Principal $\text{Diff}_{\text{vol}}(S)$ -bundle structure on volume preserving embeddings

5.1 The case $\dim S = \dim M$

We recall that in this case we require $\partial S \neq \emptyset$. We fix volume forms μ and μ_M on S and M , respectively, and we define the set of volume preserving embeddings

$$\text{Emb}_{\text{vol}}(S, M) := \{f \in \text{Emb}(S, M) \mid f^* \mu_M = \mu\}.$$

The projection $\pi(f) = f(S)$ restricted to $\text{Emb}_{\text{vol}}(S, M)$ takes values in the nonlinear Grassmannian $\text{Gr}_0^S(M)$ of all type S submanifolds of M of same volume as S , introduced in (3.3), since $f(S)$ has the same volume as S for all $f \in \text{Emb}_{\text{vol}}(S, M)$.

Using a pullback construction, we show below that $\text{Emb}_{\text{vol}}(S, M)$ has a principal $\text{Diff}_{\text{vol}}(S)$ -bundle structure over $\text{Gr}_0^S(M)$.

Proposition 5.1 *The pullback bundle of the principal bundle $\pi_{\text{vol}} : \text{Emb}(S, M) \rightarrow \text{Gr}^{S, \mu}(M)$ (4.8) with structure group $\text{Diff}_{\text{vol}}(S)$, via the smooth section*

$$\eta : \text{Gr}_0^S(M) \rightarrow \text{Gr}^{S, \mu}(M), \quad \eta(N) = (N, i_N^* \mu_M)$$

of the $\text{Vol}_\mu(S)$ bundle $\pi_1 : \text{Gr}^{S,\mu}(M) \rightarrow \text{Gr}^S(M)$ (4.3), can be identified with

$$\pi : \text{Emb}_{\text{vol}}(S, M) \rightarrow \text{Gr}_0^S(M).$$

The tangent space is $T_f \text{Emb}_{\text{vol}}(S, M) = \{v_f \in \Gamma(f^*TM) \mid \text{div}_{\mu_M}(v_f \circ f^{-1}) = 0\}$.

Proof. The condition $\eta(N) = \pi_{\text{vol}}(f)$ is equivalent to $f(S) = N$ and $f^*\mu_M = \mu$. Hence the pullback bundle

$$\eta^* \text{Emb}(S, M) := \{(N, f) \in \text{Gr}_0^S(M) \times \text{Emb}(S, M) : \eta(N) = \pi_{\text{vol}}(f)\}$$

can be identified with $\text{Emb}_{\text{vol}}(S, M)$, while the canonical projection on $\text{Gr}_0^S(M)$ becomes $\pi : f \mapsto f(S)$. ■

As a consequence we also get a natural Fréchet manifold structure on $\text{Emb}_{\text{vol}}(S, M)$.

5.2 The case $\dim S < \dim M$

In addition to a volume form μ on S (with or without boundary), we fix a Riemannian metric g on M . As in Section 3.2, we denote by $\mu(g)$ the Riemannian volume form on M and by $\mu(g_N)$ the Riemannian volume form induced by the Riemannian metric g restricted to N . In this case we define

$$\text{Emb}_{\text{vol}}(S, M) := \{f \in \text{Emb}(S, M) \mid f^*\mu(g_{f(S)}) = \mu\}.$$

and

$$\eta : \text{Gr}_0^S(M) \rightarrow \text{Gr}^{S,\mu}(M), \quad \eta(N) := (N, \mu(g_N))$$

so we have

$$\eta^* \text{Emb}(S, M) := \{(N, f) \in \text{Gr}_0^S(M) \times \text{Emb}(S, M) : \eta(N) = \pi_{\text{vol}}(f)\},$$

which can be identified with $\text{Emb}_{\text{vol}}(S, M)$. This proves that $\text{Emb}_{\text{vol}}(S, M)$ is a Fréchet manifold and the total space of a principal $\text{Diff}_{\text{vol}}(S)$ -bundle over $\text{Gr}_0^S(M)$, for $\dim S < \dim M$ and S with or without boundary. The tangent space is

$$T_f \text{Emb}_{\text{vol}}(S, M) = \{v_f \in \Gamma(f^*TM) : \text{div}_{\mu(g_N)}(v_f \circ f^{-1})^\top = g(H_N, v_f \circ f^{-1})\}.$$

We notice that the principal bundle

$$\pi : \text{Emb}_{\text{vol}}(S, M) \rightarrow \text{Gr}_0^S(M)$$

is a reduction of the structure group of the principal bundle $\text{Emb}_0(S, M) \rightarrow \text{Gr}_0^S(M)$ from $\text{Diff}(S)$ to the subgroup $\text{Diff}_{\text{vol}}(S)$.

References

- Balleier, C. and T. Wurzbacher [2012], On the geometry and quantization of symplectic Howe pairs, *Math. Z.* **271**, 577–591.
- Binz, E., and H. R. Fischer [1981], The manifold of embeddings of a closed manifold, with an appendix by P. Michor, in *Differential geometric methods in mathematical physics* (Proc. Internat. Conf., Tech. Univ. Clausthal, 1978), Lecture Notes in Phys. **139**, Springer, Berlin, 1981, 310–329.
- Ebin, D. G. and J. E. Marsden [1970], Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92**, 102–163.
- Gallot, S., D. Hulin, J. Lafontaine [2004], *Riemannian Geometry*, Springer-Verlag - Universitext, third edition.
- Gay-Balmaz, F., J. E. Marsden, and T. S. Ratiu [2012], Reduced variational formulations in free boundary continuum mechanics, *J. Nonlin. Sci.*, **22**(4), 463–497.
- Gay-Balmaz, F. and C. Vizman [2013], A dual pair for free boundary fluids, preprint.
- Gay-Balmaz, F. and C. Vizman [2014], Isotropic submanifolds and coadjoint orbits of the Hamiltonian group, preprint
- Gloeckner, H. [2006], Implicit functions from topological vector spaces to Banach spaces, *Israel Journal Math.*, **155** 205–252.
- Haller, S. and C. Vizman [2004], Non-linear Grassmannians as coadjoint orbits, *Math. Ann.* **329**, 771–785.
- Hamilton, R. S. [1982], The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.*, **7**(1), 65–222.
- Hirsch, M. W. [1976], *Differential topology*, Graduate Texts in Math. **33**, Springer.
- Kriegl, A., and P. W. Michor [1997], *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs **53**, American Mathematical Society, Providence, RI.
- Lewis, D., J. E. Marsden, R. Montgomery, and T. S. Ratiu [1986], The Hamiltonian structure for dynamic free boundary problems, *Physica D* **18**, 391–404.
- Michor, P. W. [1980], *Manifolds of Differentiable Mappings*, Shiva Publishing Limited.
- Marsden, J. E. and A. Weinstein [1983], Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids, *Phys. D* **7**, 305–323.
- Molitor, M. [2008], La Grassmannienne non-linéaire comme variété Fréchetique homogène, *J. Lie Theory* **18**, 523–539.
- Molitor, M. [2012], Remarks on the space of volume preserving embeddings, arXiv:1204.3155.
- Neeb, K.-H. and F. Wagemann [2008], Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds, *Geometriae Dedicata* **134**, 17–60.

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